QUEST FOR 5D POLYTOPES USING A GENETIC ALGORITHM

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Abstract. Five-dimensional polytopes play a role in string theory, especially when they are reflexive. It is not trivial to find reflexive polytopes and there exists genetic approaches to finding them. We are implementing a genetic algorithm in Python and have found new five-dimensional polytopes.

1. Introduction

Before we start with the implementation and the actual polytop search, we want to explain a few basics.

1.1. **What is a polytope?** First of all, when we talk about polytopes, we mean lattice polytopes. In mathematics, a lattice in Euclidean space is a regular grid of points stretching infinitely in all directions. Imagine a 3D grid where points are spaced evenly in all three dimensions. These points have integer coordinates. In two dimensions, it would look like a regular grid of dots on a piece of graph paper.

A polytope is a general term that encompasses geometric shapes in any number of dimensions. In two dimensions, a polytope is a polygon (like a square or triangle), in three dimensions, it's a polyhedron (like a cube or pyramid), and in higher dimensions, we simply call them polytopes.

A lattice polytope is a polytope whose vertices are all located at the points of a lattice. That is, the vertices of a lattice polytope is integer-valued.

More formally, a d-dimensional lattice polytope P is the convex hull in $\mathbb{R}^d$ of a finite number of lattice points $x_1, \ldots, x_m \in \mathbb{Z}^d \subset \mathbb{R}^d$. We can list these points in the format of an $d \times m$ matrix $X = (x_1, \ldots, x_m)$ whose columns are the generators.

The first important quantity is the set of vertices $\{v_1, \cdots, v_{N_v}\}$ where the number of such vertex points denotes $N_v(P) \leq m$. It is less than or equal to the number of lattice points of the polytope as one could have, for example, three colinear lattice points the middle one not being included in the vertex set. We can combine these vertices into an $d \times N_v$ matrix $V = (v_1, \cdots, v_{N_v})$ called the vertex matrix.

2010 *Mathematics Subject Classification.* 52B05.

Key words and phrases. Polytopes, Calabi-Yau Manifold, Genetic Algorithm.

Now, define $H = \{x \in \mathbb{R}^d \mid u \cdot x = b\}$ be a hyperplane where $u \in \mathbb{Z}^d$ is a primitive inner normal and $b \in \mathbb{Q}$ is some rational number ("Hesse normal form" [1, p. 28,71], [2, p. 369]). "Primitive" means the coordinates of the normal are coprime [3] (comparable to the concept of an unit vector). "Inner" means that the normal is directed towards the inside of the polytope, meaning in the direction of the origin. We call a hypersurface valid if the polytope P is contained in the negative half-space (if $u \cdot x \leq b$ for all $x \in P$).

We call a face of P to be the intersection of P with all valid hyperplanes H. A facet is a face of dimension $d-1$.

1.2. **What is the lattice distance?** The lattice distance between a lattice hyperplane $H \subset \mathbb{R}^d$ and a lattice point $x \in \mathbb{Z}^d$ is $\operatorname{dist}(x, H) \coloneqq |f(x)|$, where f is a primitive functional with $f(H) = 0$ [4, p. 50].

We can also say, the lattice distance $\operatorname{dist}(x, F)$ of a lattice point x of a polytope P from a facet F of P is defined as $\langle u, x\rangle - b = u \cdot x - b$, where $u \in \mathbb{Z}^d$ is a primitive inner normal of F and the hyperplane H spanned by F is given as $H = \{x \in \mathbb{R}^d \mid u{\cdot}x = b\}$ [5].

1.3. **What makes a lattice polytope reflexive?** The reflexive property of a polytope is given if the polytope has only one interior point and the lattice distance of each polytope's facet to this interior point is 1.

Hence we want polytopes with the following two properties

- The Interior Point (IP) property. A polytope satisfies the IP property if its only interior lattice point is the origin $O = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
- The lattice distance between each facet of the polytope and its sole interior point, namely the origin, is 1.

Two polytopes P and Q with the same number of vertices are considered equivalent if their vertices are related by some integer linear transformation plus permutation matrix. An easy example to thing about is taking a square around the origin in $\mathbb{Z}^2$ and rotating it by 45 degrees. The resulting shape is a diamond around the origin but in essence its the same polytope as the original one.

Berglund, He, Heyes, Hirst, Jejjala, and Lukas [6] have demonstrated how a genetic approach can be used to identify new reflexive polytopes. The authors have found various polytopes and published their data set on GitHub [7]. We take up this idea and embark on a search for new five-dimensional polytopes.

For the quest of reflexive polytopes by genetic approach, we choose the following initial parameters :

- The dimension $d = 5$.
- The number of vertices of the polytope m. This will serve to organise points of P in a $5 \times m$ matrix.

- The value of x_{min} and ν which will serve as the range for the variables x_a^i of the matrix where $x_a^i \in \{x_{min}, \cdots, x_{min} + 2^\nu + 1\}$. This is simply a range for the size of the coordinates of the vertices of P where ν is the number of bits used for each matrix entry. This serves to define the environment E which will be the space of all $d \times m$ matrices with interval values in this range (phenotypes).

2. Implementing the genetic algorithm

To be able to get the most out of the Genetic Algorithm, the possibility of adjustments on several ends is needed. Therefore, no Genetic Algorithm Library was used. Instead, the whole structure was built up from the scratch. The chosen approach is based on previous work in the fields of Genetic Algorithms.

Genetic Algorithms are searching the whole solution space through applying the same steps over and over again to a number of solutions. The terms for a Genetic Algorithm in our context are as following: Representation: As a polytope can be easily represented by a matrix, we chose such a matrix to be the representation and thus the base for our algorithm. Generation: A set number n, which describes the randomly created number polytopes. n stays always the same during the execution of the Algorithm. Population: A set number m, which describes the number of Generations the algorithm creates. Chromosome: A chromosome describes a solution of the given problem. In our case, the matrix of a polytope. Gen: A Gen describes a point in a polytope representation.

Moreover we need to be able to rate those polytope representations. The rating is based on a fitness function, which calculates a numeric value for the given chromosome. Berglund et al. [6] offer a highly performance intense function, which includes the number of interior points as well as the lattice distance to the origin. As our target is to find as many polytopes as possible, performance is a important factor. By excluding the interior point property the performance dramatically improves especially for bigger polytopes which can often appear as the representation matrix of the initial polytopes are created randomly.

Moreover the implemented fitness function has a second step, that apply after finding a reflexive polytope. In that case, to make sure not all reflexive polytopes have the same fitness, we chose bigger polytopes to be more valuable that smaller ones. Therefore two options are in place: - adding up the absolute Value of every single number in the given representation matrix. - Take the difference between the biggest and the smallest number given in the representation matrix.

The algorithm moreover uses a random one point crossover and simple mutation functions to systematically evolve the given solution space.

After defining basic information like the number of Generations, the number of Chromosomes per Generation, the dimension of the polytope and upper and lower boundary for the random polytope creation, the Algorithm is ready to use and provides solutions depending on the set constants.

3. Two coordinates make the race

We identified that 2 coordinates form a major adjustment screw.

4. Results

With the given Algorithm we were able to find new and way bigger polytopes than identified by now for example by [7] as of 2024-01-01.

The two matrices, see equation (1) and equation (2), each contain the 9 vertices of newly identified polytopes that have a coordinate difference of 19 and thus are substantially larger than those known so far.

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & -3 & -1 & 3 \\ -1 & 2 & 1 & 0 & -1 & -2 \\ 1 & -2 & 0 & 2 & 1 & 1 \\ -1 & 1 & -1 & -7 & -2 & 5 \\ -5 & -69 & -4 & 0 & 0 & 141 \end{pmatrix} \tag{1}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & -3 & -1 & 3 \\ -1 & 2 & 1 & 0 & -1 & -2 \\ 1 & -2 & 0 & 2 & 1 & 1 \\ -1 & 1 & -1 & -7 & -2 & 5 \\ -5 & -99999 & -4 & 0 & 0 & 200001 \end{pmatrix} \tag{2}$$

The interior point of both polytopes is $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Figure 1 displays these two newly found polytopes, projected from 5D into 3D.

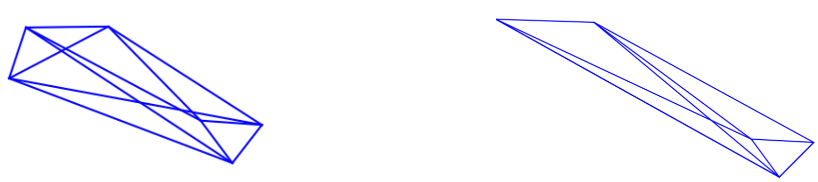

Figure 1. Newly found polytopes, the one given by matrix (1) on the left and the other one given by matrix (2) on the right

The reflexivity feature of this polytope can be verified using Sage as demonstrated by Listing 1 or using the CYTools framework [8] as shown by Listing 2, both given in the appendix A. The complete code is available at GitHub [9].

5. Conclusion and outlook

In the next steps, we optimize the genetic search algorithm and increase the search space to larger polytopes, that is, we expand the search for polytopes that have larger coordinates.

Another next step is evolving the visualization framework to provide a better and unambiguous graphical representation of the results. Barnette [10] as well as Wang, Yu, Chung, Gdawiec, and Ouyang [11], for instance, provide promising approaches that we could build on.

Appendix A. Listings

```
p = LatticePolytope([(-2, 3, 1, -3, 0), (1, -2, -1, 2, 0), (4, -4, 1,
   2, -2), (1, 2, 3, 0, -1), (-3, 3, -1, -2, 2), (1, 2, 3, -3, -1),
   (-2, -1, -3, 0, 2)])
p.is_reflexive()
```

Listing 1. Verify the polytope's reflexivity using Sage

```
from cytools import Polytope
vertices = [[-2, 3, 1, -3, 0],
 [1, -2, -1, 2, 0],
 [4, -4, 1, 2, -2],
 [1, 2, 3, 0, -1],
 [-3, 3, -1, -2, 2],
 [1, 2, 3, -3, -1],
 [-2, -1, -3, 0, 2]]
p = Polytope(vertices)
print(p.is_reflexive())
```

Listing 2. Verify the polytope's reflexivity using CYTools

```
p = Polytope(vertices)
print(p.hpq(1,1,lattice="N"))
print(p.hpq(1,2,lattice="N"))
print(p.hpq(1,3,lattice="N"))
print(p.hpq(2,2,lattice="N"))
print(p.chi(lattice="N"))
```

Listing 3. Retrieve the Hodge number $h^{p,q}$ and the Euler characteristic using CYTools

We used the function `IsIsomorphic(P,Q) : TorPol,TorPol -> BoolElt, Map` in Magma Computational Algebra System to check whether two polytopes P and Q are isomorphic, that is if there exists an element in $GL_n(Z)$ sending P to Q.

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